The Family Of Adi Polytopes

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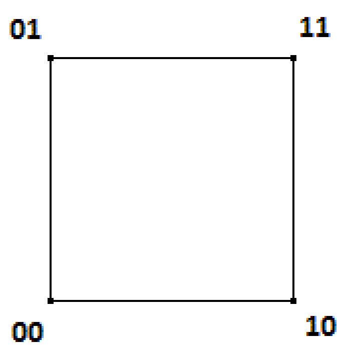
Introduction

This paper explains what an adi polytope is. There is one adi polytope in every dimension of space. I shall use inductive reasoning to show this by working up through the dimensions:

2 Space

A trivial example of an adi polytope is the line ab. This is the adi polytope in 2 dimensional space.

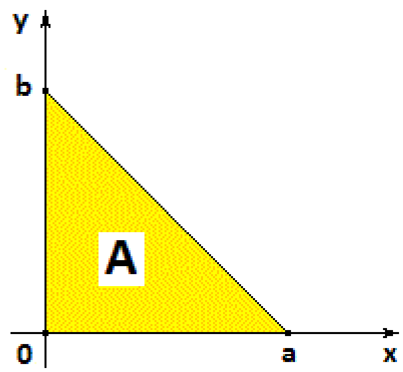


Squares, cubes and hypercubes are used to find adi polytopes. Using binary to connect the vertices of a square in two dimensional space, all two digit binary numbers are connected if and only if there is one digit different.

In 2 space a simplex is represented as a triangle.

$$Area\ A = \frac{1}{2}\ base\ X\ height$$

$$= \frac{ab}{2}$$



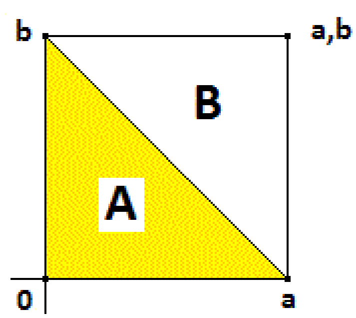
Using integration to check that half base times height equals area A:

$$Area\ A = \int_{x=0}^{x=a} \int_{y=0}^{y=b-\frac{b}{a}x} 1\ \delta y.\delta x.$$

$$= \int_{x=0}^{x=a} b - \frac{b}{a}x\ \delta x$$

$$= ab - \frac{ab}{2}$$

$$=>\ A\ =\ \frac{ab}{2}$$

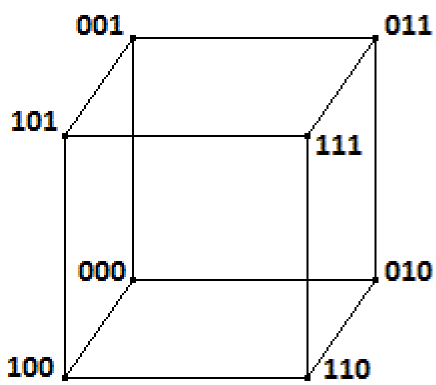


Area A = Area B

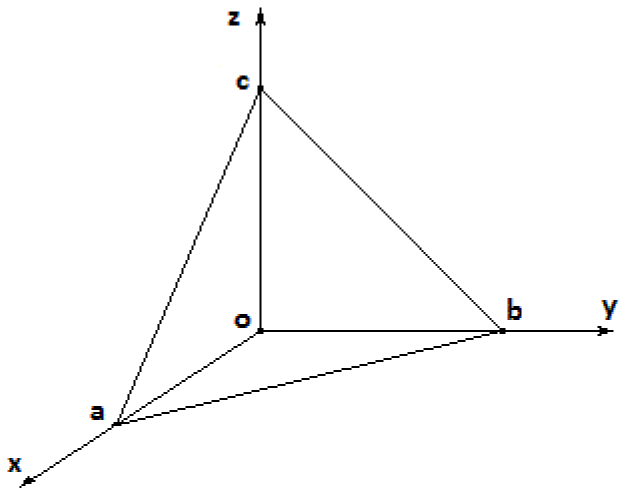
The 2 space adi polytope area = a.b-(A+B) = 0

3 Space

The example of an adi polytope in three dimensional space is the tetrahedron. We can get this tetrahedron from a cube.



Using binary to connect the vertices of a cube in three dimensional space, all three digit binary numbers are connected if and only if there is one digit different.



$$volume\,V \;=\; \frac{1}{3}\,base\;X\;height$$

$$= \frac{1}{3}\,.\,\frac{ab}{2}\,.\,c$$

$$= \frac{abc}{6}$$

Using integration to check that one third base times height equals volume V:

$$V \;+\; \int_{x=0}^{x=a}\int_{y=0}^{y=b-\frac{b}{a}x}\int_{z=0}^{z=c-\frac{c}{b}y-\frac{c}{a}x} 1\;\delta z.\delta y.\delta x. \;=\; 0$$

$$=> \;V \;+\; \int_{x=0}^{x=a}\int_{y=0}^{y=b-\frac{b}{a}x} c-\frac{c}{b}y-\frac{c}{a}x \quad \delta y.\delta x. \;=\; 0$$

$$=> \;V \;+\; \int_{x=0}^{x=a} c\left(b-\frac{b}{a}x\right)-\frac{c}{2b}\left(b-\frac{b}{a}x\right)^2-\frac{c}{a}x\left(b-\frac{b}{a}x\right)\delta x \;=\; 0$$

$$=> \;V \;+\; \int_{x=0}^{x=a} c\left(b-\frac{b}{a}x\right)-\frac{c}{2b}\left(b-\frac{b}{a}x\right)\left(b-\frac{b}{a}x\right)-\frac{c}{a}x\left(b-\frac{b}{a}x\right)\delta x \;=\; 0$$

$$=> \;V \;+\; \int_{x=0}^{x=a} c\left(b-\frac{b}{a}x\right)-\frac{c}{2b}\left(b^2+\frac{b^2}{a^2}x^2-2\frac{b^2}{a}x\right)-\frac{c}{a}x\left(b-\frac{b}{a}x\right)\delta x \;=\; 0$$

$$=> \;V \;+\; \int_{x=0}^{x=a} c\left(b-\frac{b}{a}x\right)-\frac{b^2c}{2b}-\frac{b^2c}{2a^2b}x^2+\frac{2b^2c}{2ab}x-\frac{c}{a}x\left(b-\frac{b}{a}x\right)\delta x \;=\; 0$$

$$=> \ V \ + \ \int_{x=0}^{x=a} c\left(b - \frac{b}{a}x\right) - \frac{bc}{2} - \frac{bc}{2a^2}x^2 + \frac{bc}{a}x - \frac{c}{a}x\left(b - \frac{b}{a}x\right)\delta x \ = \ 0$$

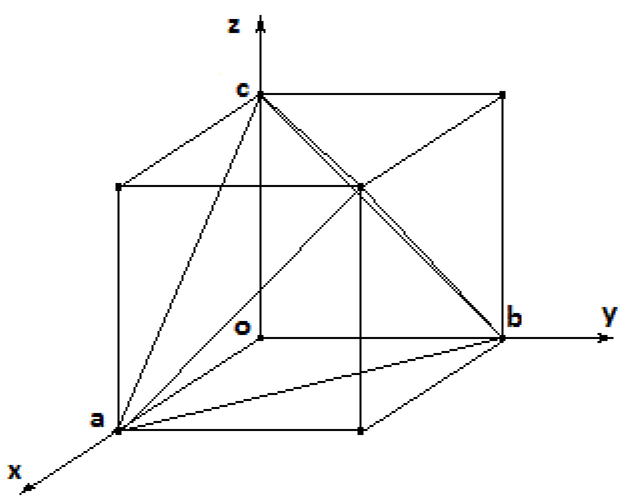
$$=> \ V \ + \ \left[c\left(bx - \frac{b}{2a}x^2\right) - \frac{bc}{2}x - \frac{bc}{6a^2}x^3 + \frac{bc}{2a}x^2 - \frac{c}{a}x\left(bx - \frac{b}{a}x^2\right)\right] \ = \ 0$$

$$=> \ V \ + \ \left(abc - \frac{a^2bc}{2a} - \frac{abc}{2} - \frac{a^3bc}{6a^2} + \frac{a^2bc}{2a} - \frac{a^2bc}{a} + \frac{a^3bc}{2a^2}\right) \ = \ 0$$

$$=> \ V \ + \ \left(abc - \frac{abc}{2} - \frac{abc}{2} - \frac{abc}{6} + \frac{abc}{2} - abc + \frac{abc}{2}\right) \ = \ 0$$

$$=> \ V \ = \ \frac{abc}{6}$$

The adi polytope in 3 space is the tetrahedron T0{100, 010, 001, 111}



There are four right tetrahedra:

T1 {000, 100, 010, 001}

T2 {100, 101, 001, 111}

T3 {010, 011, 001, 111}

T4 {100, 010, 110, 111}

The volume V of each of the four tetrahedra above is:

$$V = \frac{abc}{6}$$

So the volume V of the tetrahedron within the cube T0{100, 010, 001, 111} is:

$$= abc - \frac{4abc}{6}$$

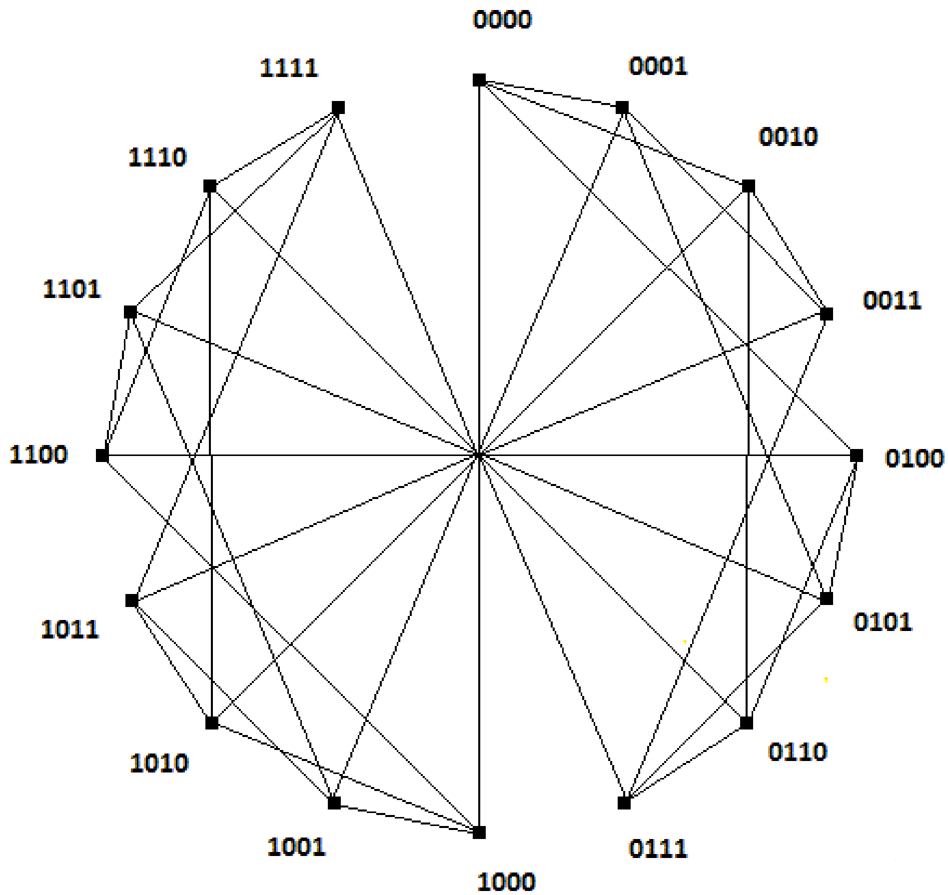
$$=> V = \frac{abc}{3}$$

4 Space

The vertices of the hypercube in 4 space can be represented by all four digit binary numbers:

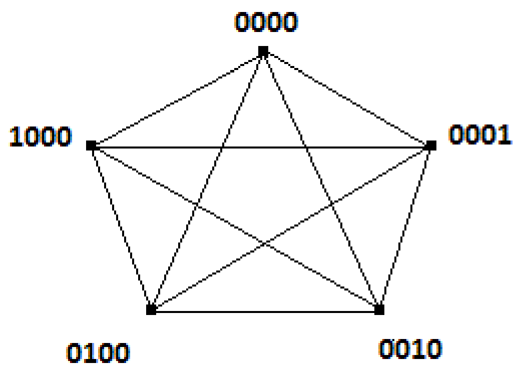
4HC {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}

All four digit binary numbers are connected if and only if there is one digit different. The connections represent the edges of the hypercube:



There are sixteen vertices on a 4 space hypercube and so there are sixteen possible right simplexes. An example of the right simplex at vertex 0000 would be:

4RS0000 {0000, 0001, 0010, 0100, 1000}



To find the adi polytope in 4 space we only need to use half the right simplexes. If we use the RS0000 simplex then none of the other eight right simplexes use the vertex 0000.

The hypercube in 4 space:

4HC {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}

The sixteen right simplexes of the hypercube in 4 space:

4RS0000 {0000, 0001, 0010, 0100, 1000}*

4RS0001 {0001, 0000, 0011, 0101, 1001}

4RS0010 {0010, 0000, 0011, 0110, 1010}

4RS0011 {0011, 0010, 0001, 0111, 1011}*

4RS0100 {0100, 0000, 0101, 0110, 1100}

4RS0101 {0101, 0001, 0100, 0111, 1101}*

4RS0110 {0110, 0010, 0100, 0111, 1110}*

4RS0111 {0111, 0011, 0101, 0110, 1111}

4RS1000 {1000, 0000, 1001, 1010, 1100}

4RS1001 {1001, 0001, 1000, 1011, 1101}*

4RS1010 {1010, 0010, 1000, 1011, 1110}*

4RS1011 {1011, 0011, 1001, 1010, 1111}

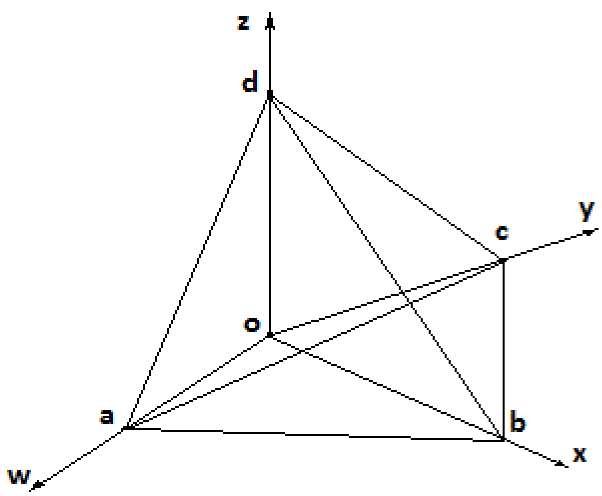
4RS1100 {1100, 0100, 1000, 1101, 1110}*

4RS1101 {1101, 0101, 1001, 1100, 1111}

4RS1110 {1110, 0110, 1010, 1100, 1111}

4RS1111 {1111, 0111, 1011, 1101, 1110}*

To find the adi polytope we need to find all the right simplexes that do not repeat the right vertices and so if we initially choose 0000 then we get those right simplexes that have an asterisk by them.



The volume V of a right simplex in 4 space is:

$$Volume\ V\ =\ \frac{1}{4}\ base\ X\ height$$

$$=\ \frac{1}{4}.\frac{abc}{6}.d$$

$$=\ \frac{abcd}{24}$$

The proof of this can be obtained by working out the following quadruple integral:

$$\int_{w=0}^{w=a}\int_{x=0}^{x=b-\frac{b}{a}w}\int_{y=0}^{y=c-\frac{c}{b}x-\frac{c}{a}w}\int_{z=0}^{z=d-\frac{d}{c}y-\frac{d}{b}x-\frac{d}{a}w} 1\ \delta z.\delta y.\delta x.\delta w$$

$$=\ \frac{abcd}{24}$$

I have not shown the working out here as it is rather lengthy.

To find the vertices of the adi polytope in 4 space we get the eight simplexes minus the right vertices. There is one in each right simplex which are the vertices in chevron brackets below:

4RS0000 {<0000>, 0001, 0010, 0100, 1000}*

4RS0011 {<0011>, 0010, 0001, 0111, 1011}*

4RS0101 {<0101>, 0001, 0100, 0111, 1101}*

4RS0110 {<0110>, 0010, 0100, 0111, 1110}*

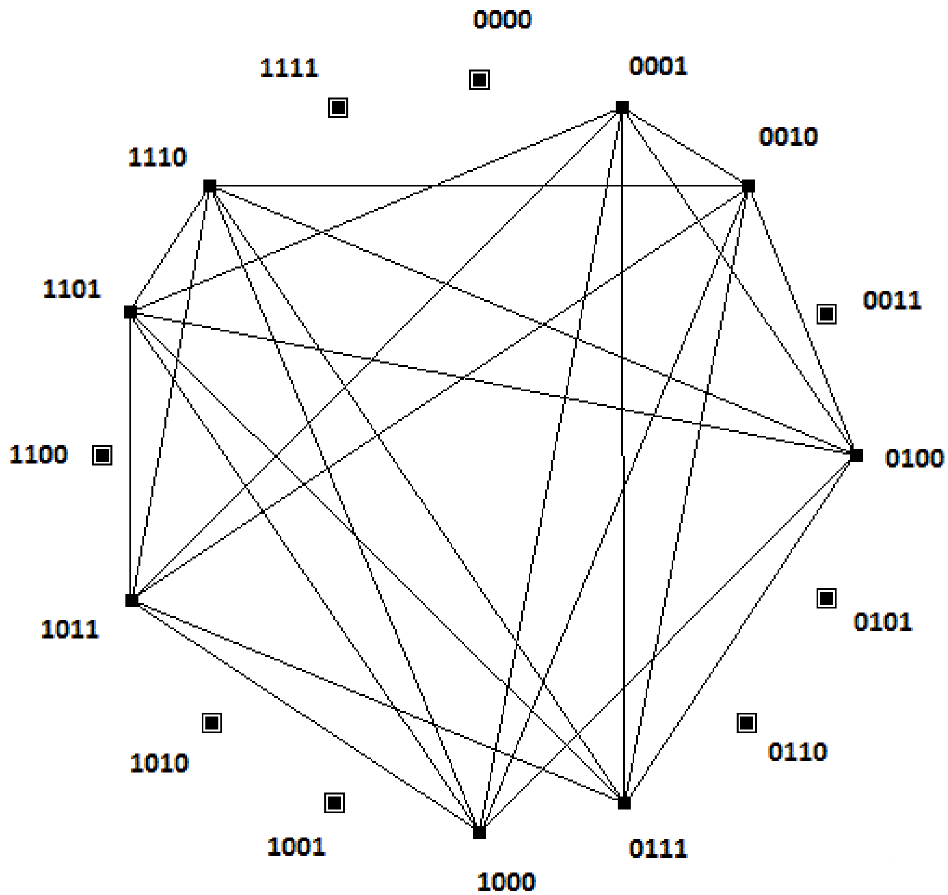
4RS1001 {<1001>, 0001, 1000, 1011, 1101}*

4RS1010 {<1010>, 0010, 1000, 1011, 1110}*

4RS1100 {<1100>, 0100, 1000, 1101, 1110}*

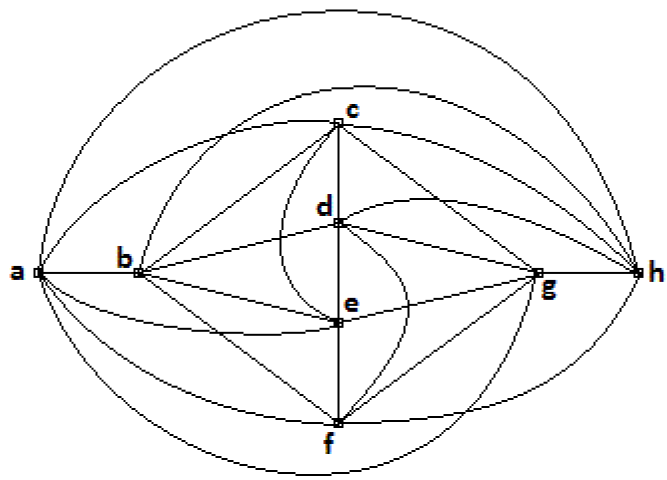
4RS1111 {<1111>, 0111, 1011, 1101, 1110}*

The following graph is what we get when we plot the points of the 4 space adi polytope:



The volume V of the adi polytope in 4 space is:

This below is proof that there is a four dimensional polytope with eight vertices each of order six. It is proof because each vertex joins six other vertices and does not cross more than one edge.



n Space

To find the n hypercube just use every n digit binary number and then connect all the edges. (Where there is only one digit different then there is an edge connection.)

To find the right n simplexes, choose any vertex then find the other n vertices which only have one digit different, so that each right simplex set has n + 1 vertices, all of which connect to every other vertex in the set.

The volume V of a regular simplex is:

$$V = \frac{x^n}{n!}$$

Where n is the dimension and x is the length of the edges.

Question:

What is the value of edge length x in all dimensions, where the volume of the regular simplex equals one?

Answer:

$$\frac{x^n}{n!} = 1 \Rightarrow x^n = n!$$

$$=> \; n \, ln \, x \; = \; ln\,(n!)$$

$$ln \, x \; = \; \frac{ln\,(n!)}{n}$$

$$x \; = \; e^{\frac{ln(n!)}{n}}$$

Check: in the case where n = 3.

$$x \; = \; e^{\frac{ln(3!)}{3}}$$

$$= \; e^{\frac{ln(6)}{3}}$$

$$= \; 1.817120593 \; to \, 9 \; d.p.$$

We know that a regular tetrahedron has volume

$$V \; = \; \frac{x^3}{6} \; = \; 1$$

where V = 1

$$x^3 = 6$$

$$x = 6^{\frac{1}{3}}$$

$$= \; 1.817120593 \; to \, 9 \; d.p.$$

$$therefore \quad e^{\frac{ln(n!)}{n}} \equiv (n!)^{\frac{1}{n}}$$

Adi polytopes in n space:

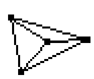
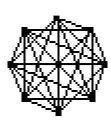
Number of vertices p $$p = 2^{n-1}$$

Vertex order r where n < 0 $$r = \frac{(2(n-1)+1)^2 - 1}{8}$$

(Triangular numbers)

Volume v $$v = \left[1 - \frac{2^{(n-1)}}{n!}\right] x^n$$

(For regular polytopes with edge lengths x)

Dimension n	Adi Polytope	Number of vertices p	Vertices order r	Volume v
1		$2^0 = 1$	0	$x - x = 0$
2		$2^1 = 2$	1	$x^2 - \frac{2x^2}{2} = 0$
3		$2^2 = 4$	3	$x^3 - \frac{4x^3}{6} = \frac{x^3}{3}$
4		$2^3 = 8$	6	$x^4 - \frac{8x^4}{24} = \frac{2x^4}{3}$
5		$2^4 = 16$	10	$x^5 - \frac{16x^5}{120} = \frac{13x^5}{15}$
6		$2^5 = 32$	15	$x^6 - \frac{32x^6}{720} = \frac{43x^6}{45}$
7		$2^6 = 64$	21	$x^7 - \frac{64x^7}{5040} = \frac{4x^7}{315}$

8	$2^7 = 128$	28	$x^8 - \frac{128x^8}{40320} = \frac{x^8}{315}$
9	$2^8 = 256$	36	$x^9 - \frac{256x^9}{362880} = \frac{2x^9}{2835}$
10	$2^9 = 512$	45	$x^{10} - \frac{512x^{10}}{3628800} = \frac{2x^{10}}{14175}$

www.adimath.webs.com